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Semiconvection as the occasional breaking of weakly amplified internal waves

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Summary. An analysis is made of the semiconvective zones which arise in the evolution of many stars. Despite the presence of a strongly stabilizing solute gradient, growing overstable modes are possible for a slightly superadiabatic temperature gradient, because heat diffusion greatly exceeds solute diffusion or molecular viscosity. These modes are essentially identical to weakly amplified internal waves (or hydromagnetic waves if rotation and magnetic field are present). Their amplification is balanced by a cascade to higher wavevectors (caused in part by subharmonic parametric instabilities) leading to infrequent wavebreaking 'events' which provide the required redistribution of solute. Detailed quantification of this model is impractical, but a simplified analysis indicates that the ratio of the superadiabaticity to the solute gradient is at most of order $(D_e/\kappa)^{1/2}$ where D_e is the solute 'eddy' diffusivity and κ is the thermal diffusivity. Evolutionary models require $D_e \ll \kappa$, so the Schwarzschild–Härm criterion for semiconvection is essentially correct. A field in excess of about 10^4 gauss modifies the model somewhat, but does not invalidate it. Propagation of the waves out into the radiative envelope of the star is unimportant. The related phenomenon to semiconvection which occurs in differentiating black dwarfs (e.g. Jupiter, Saturn) is also briefly discussed.

1 Introduction

In stellar evolution calculations, one frequently encounters a situation in which an inhomogeneous layer forms where almost all the heat flux is transported by radiation, but where slow convection is required to redistribute a stably stratified solute (such as helium). If the work done in redistributing the solute is several orders of magnitude less than the total heat flux, then it is reasonable to suppose that the layer maintains itself in a state that is close to convective neutrality. However, a controversy has arisen concerning the correct definition

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of convective neutrality. Does it mean that the temperature gradient is just slightly superadiabatic, or does it mean that the temperature gradient is sufficiently superadiabatic to overcome the stabilizing effect of the solute gradient?

The former criterion (marginal superadiabaticity) is known as the Schwarzschild–Härm (SH) criterion, since it was used by them in their pioneering analysis of the evolution of massive stars (Schwarzschild & Härm 1958). They found that for masses in excess of about $10 M_\odot$, the helium-rich convective core could not be matched directly to a hydrogen-rich radiative exterior, because the dominance of electron scattering in the opacity meant that the exterior was more convectively unstable than the core. However, the initiation of convection in the exterior region mixes helium upwards, reducing the opacity and tending to shut-off the convection. SH proposed an intermediate zone of convective neutrality in which there is a stabilizing helium gradient, and called this self-regulating process 'semiconvection'. They chose to fix the temperature gradient at the adiabatic value, and this choice has been used frequently in subsequent studies (see Stothers & Chin 1976, and references therein).

The need for a semiconvective layer is seldom in dispute, but many other studies (e.g. Sakashita & Hayashi 1958; Stothers 1966; Stothers & Chin 1975) have been based on an alternative criterion, originally proposed (in a different context) by Ledoux (1947), which states that the temperature gradient must be highly superadiabatic to overcome the stabilizing influence of the compositional gradient. The choice of criterion affects the predicted Hertzsprung–Russell diagram for supergiants (Stothers & Chin 1976), the horizontal branch evolution of low-mass stars (Schwarzschild & Härm 1969; Sweigart & Gross 1976) and the degenerate cooling of differentiating black dwarfs such as Jupiter or Saturn (Stevenson & Salpeter 1977b) so it is of considerable importance to establish the correct description of this phenomenon.

Semiconvection is evidently related to the extensively studied thermosolutal convection (Turner 1973) which is readily studied in the laboratory, and appears naturally in the Earth's oceans. A feature of thermosolutal convection is that spontaneous fluid motion can occur even when the system is statically stable (i.e. even when the Ledoux criterion for semiconvection is not satisfied). These motions are possible because there are (at least) two density-affecting properties with different diffusivities. In both the stellar and oceanographic cases, the two components are heat and solute, where the solute is usually helium in the stellar case and salt in the oceanographic case. If the stabilizing component has the greater diffusivity, then monotonically growing instabilities called 'salt fingers' (Stern 1960; Turner 1964) are possible. In stars and oceans the solute has the smaller diffusivity, and growing oscillations are possible (Gershuni & Zhukovitskii 1963; Shirtcliffe 1967) if the solute is stabilizing.

Spiegel (1969) appealed to this laboratory and oceanographic evidence, and concluded that the SH criterion of semiconvection is most nearly correct. There are, however, some important differences between stars and terrestrial analogues, the most important being that the Prandtl number is about eight orders of magnitude smaller in stars (excepting degenerate dwarfs) than in water. The Prandtl number Pr is defined as

$$Pr \equiv \nu/\kappa \tag{1}$$

where ν is the kinematic viscosity and κ is the thermal diffusivity. The ratio of diffusivities $\tau \equiv D/\kappa$, where D is the solute diffusivity, is also very small in stars (and comparable to Pr). Stellar evolution also requires a specified solute flux which is several orders of magnitude greater than that which can be transported by diffusion, a constraint that is usually not imposed in terrestrial analogues.

The purpose of this paper is to demonstrate that the available oceanographic, laboratory and theoretical evidence *cannot* be directly applied to semiconvection, primarily because of the large difference in *Pr*. There may nevertheless be an analogy between semiconvection and vertical mixing processes in the Earth's oceans, and this is the theme that will be developed in Section 3.

We begin in Section 2 by considering the onset of convection, using linear stability analysis. Assuming $\tau \ll 1$ and $Pr \ll 1$, it is shown that growing oscillations are possible which have a dispersion relationship essentially identical to that of internal waves. Both rotation and magnetic field are included in the analysis, so both 'fast' (inertial) and 'slow' (magnetic) modes are predicted. The compositional dependence of the thermal diffusivity can significantly affect convective onset, but the required superadiabaticity is predicted to be small for all cases of interest, consistent with the SH criterion for semiconvection.

In Section 3, the previous laboratory and theoretical finite amplitude results are discussed. In thermosolutal convection (as in many convective systems) the finite amplitude state need not resemble the state predicted by linear stability analysis. In terrestrial systems, the finite amplitude state usually consists of well-mixed layers (in which there is essentially steady convection) separated by thin diffusive layers in which the solute concentration is high. It is shown that this state may not be steady if $Pr \ll 1$, since the interfacial diffusive layers can then be themselves unstable. Nevertheless, interfaces can be continuously created by the breaking of the weakly amplified internal waves. This wavebreaking preferentially occurs at wavelengths smaller than those which grow most rapidly, and occurs either as shear instabilities or as local convective instabilities. The small wavelength modes are fed by a non-linear (resonant) 'cascade' from the large wavelength modes. Since the system is only weakly turbulent, linear stability analysis can adequately approximate the growth and behaviour of the large wavelength modes. A model devised by Garrett & Munk (1972) for wavebreaking and vertical mixing in the oceans is adapted to semiconvection. For plausible choices of the energy spectrum for the waves, it is concluded that only a small superadiabaticity is required, provided the rate at which work is done in redistributing solute is much less than the total heat flux. This conclusion is insensitive to the details of the model, and can be reached by independent dimensional or mechanistic arguments. A large magnetic field might invalidate the conclusion, but this is improbable. Propagation of internal waves out of the semiconvective zone and into the outer radiative envelope occurs but is always a small sink of wave energy relative to the dominant sink of wavebreaking. No attempt is made to model the regions of transition between full convection and semiconvection and pure radiation, but it is argued that these transition regions are much thinner than the total depth of a well-developed semiconvective zone, so the dynamics of the transitions are unimportant.

In Section 4, the model is applied to actual stellar evolutions. Since the quantitative aspects of wavebreaking are poorly understood, even in the very accessible laboratory and oceanographic systems, only a semiquantitative description is possible. If the solute transport is characterized by an eddy diffusivity D_e, then the double inequality $D \ll D_e \ll \kappa$ is usually satisfied, and the superadiabaticity (expressed as a density gradient) is then much larger than that required for the onset of convection, yet much smaller than the magnitude of the density gradient due to solute. The latter ensures that the SH criterion for semiconvection is essentially correct.

In contrast, the related process to semiconvection which occurs in differentiating black dwarfs (e.g. Jupiter, Saturn) may conform to the Ledoux criterion, because of the inadequacy of conductive heat transport.

2 Linear stability analysis

The linear stability problem for thermosolutal convection has never been adequately described in the astrophysical context, so the analysis is given here in some detail. Rotation and magnetic field are both included, since their dynamical effects are usually important in stars.

Consider an infinite fluid in which the unperturbed basic state is at rest, apart from a uniform rotation $\mathbf{\Omega}$. There is a uniform gravitation acceleration $\mathbf{g}$ and a magnetic field $\mathbf{B}$. Let $\beta_T > 0$ be the magnitude of the vertical density gradient resulting from the (destabilizing) superadiabatic temperature gradient, and let $\beta_s > 0$ be the corresponding stable, vertical solute gradient. Let S and θ be small deviations in the density due to solute and temperature respectively. The linearized Boussinesq equations for θ, S, the velocity $\mathbf{\upsilon}$, and magnetic field perturbation $\mathbf{b}$ are then (Spiegel 1971, 1972; Walin 1964)

$$\frac{\partial \mathbf{\upsilon}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{\upsilon} = -\frac{1}{\rho_0}\nabla p' + \frac{\mathbf{g}(\theta + S)}{\rho_0} + \nu \nabla^2 \mathbf{\upsilon} + \frac{(\nabla \times \mathbf{b}) \times \mathbf{B}}{4\pi\rho_0} \tag{2}$$

$$\nabla \cdot \mathbf{\upsilon} = 0 \tag{3}$$

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta - W\beta_T \tag{4}$$

$$\frac{\partial S}{\partial t} = D \nabla^2 S + W\beta_s \tag{5}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \lambda \nabla^2 \mathbf{b} + \nabla \times (\mathbf{\upsilon} \times \mathbf{B}) \tag{6}$$

where ρ_0 is the unperturbed density, p' is the hydrodynamic pressure, W is the vertical component of $\mathbf{\upsilon}$ and λ is the magnetic diffusivity. The Boussinesq assumption is valid provided we consider Fourier components of $\mathbf{\upsilon} \propto \exp(i\mathbf{k}\cdot\mathbf{r} + pt)$ such that $kH_p \gg 1$, where $k \equiv |\mathbf{k}|$ and H_p is the pressure scale height. (This inequality ensures that the acoustic modes have much higher frequencies than the modes of interest, and also enables us to treat ρ_0, $\mathbf{g}$, $\mathbf{B}$, etc., as being constant. We treat κ as being constant for the moment, but see below.)

For solutions of the form $\exp(i\mathbf{k}\cdot\mathbf{r} + pt)$, equation (2) becomes

$$(p + \nu k^2)\mathbf{\upsilon} + 2\mathbf{\Omega} \times \mathbf{\upsilon} = \frac{-i\mathbf{k}}{\rho_0} p' + \frac{\mathbf{g}}{\rho_0}\left(\frac{\beta_s}{p + Dk^2} - \frac{\beta_T}{p + \kappa k^2}\right) W - \frac{(\mathbf{k}\cdot\mathbf{B})(\mathbf{k} \times \mathbf{\upsilon}) \times \mathbf{B}}{4\pi\rho_0(p + \lambda k^2)}. \tag{7}$$

The cross-product with $\mathbf{k}$ twice over then yields the dispersion relation:

$$p + \nu k^2 + \omega_M + \frac{\omega_\Omega^2}{p + \nu k^2 + \omega_M} = -\frac{k_\perp^2}{k^2}\left[\frac{N_s^2}{p + Dk^2} - \frac{N_T^2}{p + \kappa k^2}\right]$$

$$\omega_M \equiv \omega_B^2/(p + \lambda k^2)$$

$$\omega_B^2 \equiv (\mathbf{k}\cdot\mathbf{B})^2/4\pi\rho_0$$

$$\omega_\Omega^2 \equiv (2\mathbf{\Omega}\cdot\mathbf{k})^2/k^2$$

$$N_s^2 \equiv g\beta_s/\rho_0, \quad N_T^2 \equiv g\beta_T/\rho_0 \tag{8}$$

where $k_\perp$ is the horizontal component of $\mathbf{k}$. If we specify that p is real, the growing solutions are possible for $N_T^2 > N_s^2$, which corresponds to the Ledoux criterion of semiconvection. Physically, this is the requirement that the destabilizing density effect of the temperature gradient exceed the stabilizing density effect of the solute gradient.

We seek, instead, solutions for which $N_T^2 < N_s^2$ and $p = \sigma + i\omega$, where σ, ω are real and $\sigma > 0$. The general solution for (8) must then be obtained by numerical solution of two coupled sixth-order equations in σ and ω. However, an enormous simplification occurs if we make the *weak amplification approximation*: $|\sigma| \ll |\omega|$. We also assume $|\omega| \gg \lambda k^2, \nu k^2$. Neglecting all but the zeroth and first powers of σ, the imaginary part of equation (8) then gives

$$(\omega^2 - \omega_B^2)[1 - \omega^2\omega_\Omega^2/(\omega^2 - \omega_B^2)^2] = \frac{k_\perp^2\omega^2}{k^2}\left[\frac{N_s^2}{\omega^2 + D^2k^4} - \frac{N_T^2}{\omega^2 + \kappa^2k^4}\right]. \tag{9}$$

Apart from the effects of diffusion, this is precisely the dispersion relation for hydromagnetic waves (see Roberts & Soward 1972, for a review). If both ω_B and ω_Ω are non-zero then there are two branches to the dispersion spectrum $\omega(\mathbf{k})$. For small field ($\omega_B \ll N_s$), these branches are well separated and can be characterized as a 'fast' inertial mode (for which $\omega^2 \simeq (N_s^2 - N_T^2)\, k_\perp^2/k^2 + \omega_\Omega^2 + \omega_B^2$) and a 'slow' magnetic mode (for which $\omega^2 < \omega_B^2$) respectively. If $\omega_B \gtrsim N_s$ then the two branches tend to merge and $\omega \to \omega_B$.

The real part of equation (8) gives

$$2\sigma\left[1 + \frac{\omega_B^2\omega_\Omega^2}{(\omega^2 - \omega_B^2)^2}\right] = \frac{N_T^2\kappa k_\perp^2}{\omega^2 + \kappa^2k^4} - \frac{N_s^2 D k_\perp^2}{\omega^2 + D^2k^4} - k^2\left(\nu + \frac{\lambda\omega_B^2}{\omega^2}\right)\left[1 + \frac{\omega^2\omega_\Omega^2}{(\omega^2 - \omega_B^2)^2}\right]. \tag{10}$$

The critical value of N_T^2 for the onset of overstable modes is found by setting $\sigma = 0$. Growing modes are most readily achieved at low k, where $\omega_B \ll N_s$ and the inertial and magnetic branches are well separated. Consider, first, the onset of inertial mode amplification. This is most readily achieved for $\omega_\Omega = 0$, $\omega_B \to 0$ and is given by $N_T^2(\kappa + \nu) = N_s^2(D + \nu)$. It is convenient to rewrite this in dimensionless form by defining

$$\epsilon \equiv N_T^2/N_0^2$$

$$\chi \equiv N_s^2/N_0^2$$

$$N_0^2 \equiv g/H_p. \tag{11}$$

In the situations of interest, $\chi \sim 1$. The critical superadiabaticity ϵ_{CI} for inertial modes is therefore

$$\epsilon_{CI} = \frac{(D + \nu)\,\chi}{(\kappa + \nu)}. \tag{12}$$

Since D, $\nu \ll \kappa$, it follows that $\epsilon_{CI} \ll \chi$, consistent with the SH criterion for semiconvection. If $\epsilon > \epsilon_{CI}$ then amplification occurs. The result is only strictly exact for $k \to 0$ but it is still essentially valid for $H_p^{-1} \ll k \ll (N_s/\kappa)^{1/2}$, a double inequality which can always be satisfied in the situations of interest.

The critical superadiabaticity ϵ_{CM} for magnetic modes is given by

$$\epsilon_{CM} = \frac{(D + \lambda)\,\chi}{(\kappa + \lambda)} \tag{13}$$

provided $\mathbf{k}$ can be chosen so that $\omega_B^2 \ll N_s^2 k_\perp^2/k^2 \ll \omega_\Omega^2$, $\nu \ll \lambda\omega_\Omega^2 k^2/N_s^2 k_\perp^2$, and $\omega_B^2 N_s^2 k_\perp^2/k^2\omega_\Omega^2 \gg \kappa^2 k^4$. (This can usually be achieved for $k_\perp \ll k$ and $H_p^{-1} \ll k \ll (N_s/\kappa)^{1/2}$, provided the field is not too small.) If $\lambda < \nu$, which is the usual case of interest, then $\epsilon_{CM} < \epsilon_{CI}$, and the magnetic modes are most readily excited. The appropriate stability diagram is shown in

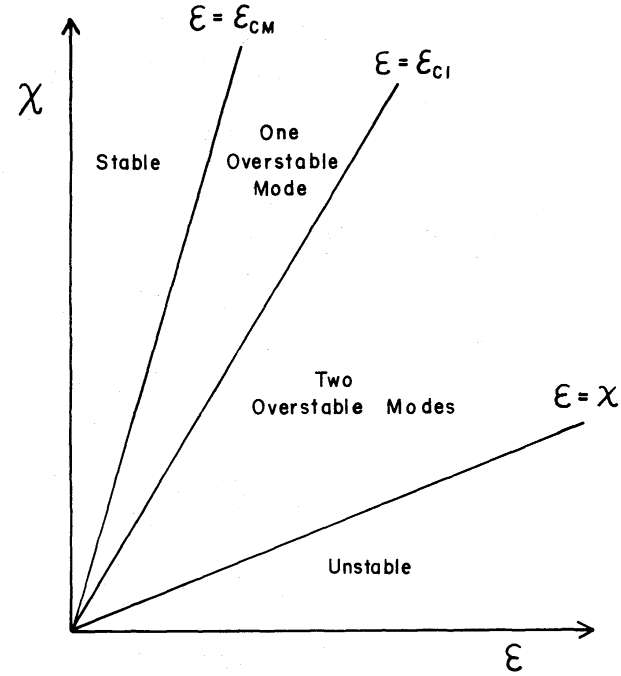

Figure 1. The linear stability diagram for thermosolutal convection in an infinite fluid in the presence o rotation and magnetic field, assuming $\chi, \epsilon > 0$ and $\kappa \gg D, \nu, \lambda$. $\epsilon_{CI} \ll \chi$ and $\epsilon_{CM} \ll \chi$ are defined by equa tions (12) and (13). The boundary between stability and overstability actually occurs near the vertica axis (consistent with the SH criterion of semiconvection) but is shown well displaced for clarity. Th boundary between overstability and instability represents the Ledoux criterion.

Fig. 1. (It is shown below that the inertial modes are nevertheless the most rapidly amplifie if $\epsilon \gg \epsilon_{CI}$.)

Since the existence of a semiconvective zone is due to the compositional dependence o the thermal diffusivity κ, it is appropriate to comment on the effect of this dependence o the convective onset. Let

$$\kappa = \kappa_0(z) + \theta \frac{\partial \kappa}{\partial \theta}(z) + S \frac{\partial \kappa}{\partial S}(z) \tag{14}$$

where z is the vertical coordinate. Equation (4) then becomes

$$\frac{\partial \theta}{\partial t} = \kappa_0 \nabla^2 \theta - W\beta_T + \frac{\partial \theta}{\partial z}\left(\frac{d\kappa_0}{dz} + \gamma_0 \frac{\partial \kappa_0}{\partial \theta}\right) + \frac{\partial S}{\partial z}\gamma_0 \frac{\partial \kappa}{\partial S} + \theta \frac{d}{dz}\left(\gamma_0 \frac{\partial \kappa}{\partial \theta}\right) + S \frac{d}{dz}\left(\gamma_0 \frac{\partial \kappa}{\partial S}\right) \tag{15}$$

where $\gamma_0(z)$ is the adiabatic temperature gradient. Since κ, γ_0 and their derivatives chang substantially over a vertical distance of order H_p, it follows that the corrections to equatio (4) can be as large as of order $(kH_p)^{-1} \kappa_0 k^2 S$. However, it is always possible to choose **k** s that only corrections of order $(kH_p)^{-2}$ occur. It then follows that $\epsilon_C \lesssim \chi/(kH_p)^2 \ll \chi$. Th critical superadiabaticity is still small, but it can be larger than that given by equation (12 Unlike κ, variability of the other 'constants' is unimportant.

The physical mechanism responsible for the growth of these overstable modes can b understood as follows. Consider the displacement of a typical fluid parcel downwards, int a warmer more helium-rich and denser environment. Since $\kappa \gg D$, thermal equilibratio proceeds much more rapidly than solute equilibration. The parcel is then lighter than i surroundings and rises. However, since κ is finite, the temperature field lags the displaceme field and the parcel returns to its initial position lighter than it was initially. It thus rises to a distance greater than its initial displacement, and a repetition of this process leads to growing oscillations. An exactly analogous process can occur in a compositionally uniform fluid but with the restoring force provided by the magnetic field (Cowling 1976).

If the superadiabaticity ϵ is much greater than the critical values (ϵ_{CI} or ϵ_{CM}) but much smaller than χ, and if dissipation is negligible, then from equation (10),

$$2\sigma\left[1+\frac{\omega_B^2\omega_\Omega^2}{(\omega^2-\omega_B^2)^2}\right]=\frac{N_T^2\kappa k_\perp^2}{\omega^2+\kappa^2k^4}. \tag{16}$$

If $\omega_B = 0$ then the maximum growth rate σ_m occurs for $\mathbf{k}$ perpendicular to both $\mathbf{g}$ and $\boldsymbol{\Omega}$ and $k = (N_s/\kappa)^{1/2}$:

$$\sigma_m = \tfrac{1}{4}(\epsilon/\chi)\,N_s. \tag{17}$$

The oscillation time is then comparable to the thermal diffusion time.

Low- and high-field regimes can be identified as $B < B_{C1}$ and $B > B_{C1}$ respectively, where

$$B_{C1} \equiv (4\pi\rho_0 N_s\kappa)^{1/2}. \tag{18}$$

If $B \ll B_{C1}$, then $\omega_B \ll N_s$ at $k = (N_s/\kappa)^{1/2}$ and the maximum growth rate of inertial modes is still given by equation (17). The most rapidly growing magnetic modes are found to have a growth rate smaller than σ_m by a factor of B/B_{C1}. If $B \gg B_{C1}$ then $\omega_B \gg N_s$ at $k = (N_s/\kappa)^{1/2}$ except for directions of $\mathbf{k}$ which satisfy $\mathbf{k}\cdot\mathbf{B}/kB \lesssim B_{C1}/B$. It is found that the maximum growth rate is still approximately σ_m, provided $\Omega \lesssim N_s$, and occurs at $\mathbf{k}\cdot\mathbf{B} = 0$. In this high-field regime, $\omega^2 \simeq \omega_B^2$ for all directions of $\mathbf{k}$ except very near $\mathbf{k}\cdot\mathbf{B} = 0$, but the most rapidly amplified modes are at $\mathbf{k}\cdot\mathbf{B} = 0$ and $\omega^2 \simeq N_s^2$. In Section 4, we tabulate values of B_{C1} and find that 10^4 gauss is a typical value. This is possibly exceeded in the interiors of many stars. Nevertheless, we shall argue that the resulting anisotropy of the wave spectrum may not play a critical role in determining the average superadiabaticity of the finite amplitude flow.

3 Finite amplitude flow

In convection problems, and especially in thermosolutal convection, the finite amplitude flow can bear little resemblance to that described by linear stability analysis (Veronis 1965; Huppert 1977). In contrast, the model proposed here for semiconvection relies heavily on the linear stability results. We justify this by showing that the available finite amplitude results are inapplicable to semiconvection, and by constructing a self-consistent (although possibly not unique) description of semiconvection in terms of small-scale wavebreaking.

Recently, Huppert & Moore (1976) have theoretically analysed the finite amplitude stability of convection confined between two planes, for specified temperature and composition changes across the fluid layer. A major finding of their work is that for a sufficiently small diffusivity ratio τ, *steady* convection can occur for a temperature gradient less than that predicted for the onset of overstability by linear stability analysis. The parameter range and geometry they considered are not appropriate to semiconvection, but the results suggest a physical picture that would at first sight appear to be relevant. The steady flow arises because of the formation of a uniformly mixed interior region, bounded at the top and bottom by diffusive layers in which most of the composition and temperature changes occur. The internal region is essentially isothermal (in the laboratory context) or adiabatic (in the semiconvection context) and one could imagine that if the boundary layers are sufficiently thin, then the average temperature gradient could be very close to adiabatic. The

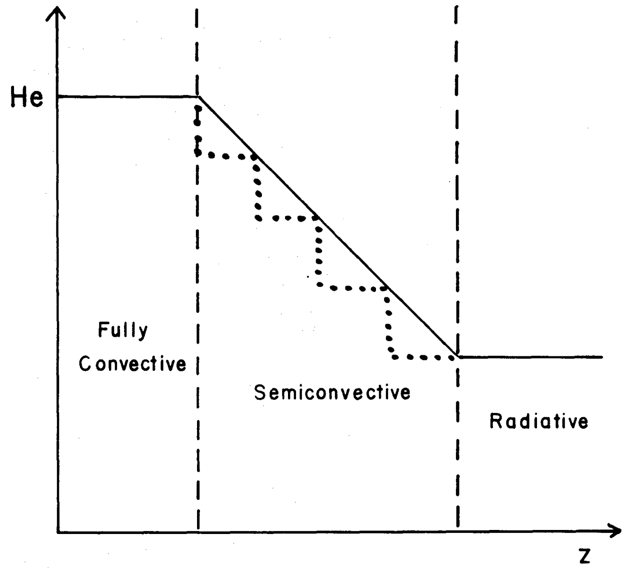

Figure 2. One possible schematic representation of helium abundance as a function of vertical distance z for a massive star. The full line represents the SH prediction, whereas the dotted line represents a hypothetical alternative in which thin diffusive regions of high helium gradient are separated by thick, homogeneous convective regions. The difficulties with this alternative are discussed in the text.

formation of thin, diffusive layers does not require the presence of a physical boundary, since a step-like distribution of solute can form spontaneously in the laboratory (Turner 1968), with the size of the steps being internally determined. In view of the apparent ubiquity of such situations, it is natural to propose this state for semiconvection. We shall now show that a simplistic application of these laboratory and oceanographic results to semiconvection is not possible, even if $B = 0$.

Consider a step-like distribution of solute, as illustrated in Fig. 2. Focus on the purely diffusive region in which most of the superadiabaticity and compositional change occur. High static stability is assumed so that entrainment of fluid (i.e. convective overshoot) across these thin diffusive regions is negligible (Linden 1974). Let the dimensionless superadiabaticity and solute gradient be ϵ_0 and χ_0, respectively, in the diffusive region. According to experiments and a simple theoretical model (Linden & Shirtcliffe 1978), the ratio of solute flux to *convective* thermal flux (both in density units) is $\tau^{1/2}$. It follows that

$$D\chi_0/\kappa\epsilon_0 = \tau^{1/2}$$

$$\epsilon_0/\chi_0 = \tau^{1/2}. \tag{19}$$

From equation (12), this diffusive layer will actually be overstable to inertial modes if $(\kappa + \nu)\epsilon_0 > (D + \nu)\chi_0$ (provided it is not too thin). This implies

$$Pr < \tau^{1/2}, \tag{20}$$

a requirement that is not satisfied in the oceans or experiments involving water (where $Pr > 1$ and $\tau < 1$) but is always satisfied in stars, and may be satisfied in experiments involving liquid metals (Jakeman & Hurle 1972). This is shown schematically in Fig. 3. It follows that the step-like distribution of solute which is ubiquitous at $Pr \gtrsim 1$ (e.g. water) need not apply to semiconvection.

A possible objection to this argument is that the interfacial regions are very thin, and one should consider the onset of overstability in a confined geometry. Using the linear stability

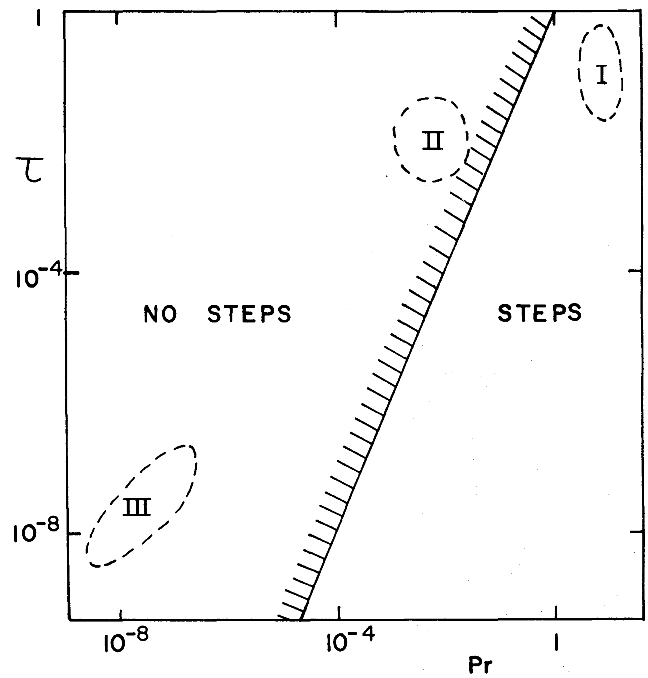

Figure 3. The finite amplitude stability diagram for a step-like distribution of solute. To the right of the line $Pr = \tau^{1/2}$, a step-like distribution is stable. The dashed line enclosing Region I indicates the parameter range encountered in oceanography and in experiments involving molecular fluids such as water. To the left of the line $Pr = \tau^{1/2}$, a step-like distribution of solute may be overstable (*cf.* equation (20)). Region II indicates the parameter range for liquid metals and liquid planet cores, while Region III corresponds to the massive stars under consideration. The hatching on the left-side of the solid line is intended to indicate that the actual criterion for no steps depends on solute flux and the depth of the layers as well as Pr and τ (see equation (21)).

results for a confined geometry (Veronis 1968), the generalization of equation (20) is

$$Pr < \tau^{1/2} - \frac{Ra_0(1+\tau)(1+\tau/Pr)(1+Pr)}{Ra_s(1-\tau^{1/2})}$$

$$Ra_s \equiv \frac{g(\Delta\rho_s/\rho_0)\,H^3}{\nu\kappa} \tag{21}$$

where $\Delta\rho_s/\rho_0$ is the fractional density drop due to solute across the diffusive layer of thickness H, and $Ra_0 \sim 10^3$ is a numerical factor which depends on the boundary conditions at the diffusive–convective boundary.

The value of H depends on the solute flux and the number of steps. Let us consider now whether a consistent solution can be constructed which avoids the difficulty presented by equation (21). In Fig. 2, a hypothetical step-like distribution of solute is shown and compared to the solute distribution predicted by SH. The steps must lie below the SH prediction because the thermal diffusivity is a decreasing function of helium concentration, so the fully convective regions between the solute steps must necessarily have less helium to ensure that some heat is transported by convection. For a stable distribution of steps, the solute flux F_S is related to the convective heat flux $F_{T,\mathrm{conv}}$ (both in density units) by $F_S \simeq \tau^{1/2} F_{T,\mathrm{conv}}$ (Linden 1974; Linden & Shirtcliffe 1978). Let ΔS be the average difference in composition between the predicted SH distribution and the mixed layer at the same vertical position in Fig. 2. A given fractional change in composition implies a comparable fractional decrease in thermal diffusivity, so there is a fraction of the total heat flux which cannot be transported by radiation along an adiabat in the mixed layer and must now be transported by convection. This fraction is of order $\Delta S/S_0$, where S_0 is the total compositional change across the semiconvective zone, so $F_{\mathrm{T,conv}} \sim (\Delta S/S_0)\, F_{\mathrm{T,total}}$. If n is the number of layers then $S_0 = n\Delta S$, so we finally conclude that

$$n \simeq \tau^{1/2} F_{\mathrm{T,total}}/F_{\mathrm{s}}. \tag{22}$$

Notice that this result is an essential consequence of the compositional dependence of κ, and has no analogue in the laboratory or ocean. By a remarkable coincidence, stellar evolutions (see Section 4) predict that n is within an order of magnitude of unity. If n is near unity, then the SH criterion would be clearly invalid (i.e. it would be more correct to model the system by a large discontinuity in helium abundance at the fully convective–semiconvective boundary). However, this solution is not self-consistent, since the substitution of realistic parameter values into equation (21) indicates that the thin diffusive layers would be unstable. (The appropriate parameters are described in Section 4.)

It follows that if the Prandtl number is sufficiently small, then a stable step-like distribution of solute is not possible. This argument does not, however, disprove the existence of layering, since the steps can migrate with time, coalesce, or be dynamically renewed; processes which occur in laboratory systems and oceans (Huppert 1977). The following heuristic argument suggests that the likely situation in semiconvection is a state that is very close to the SH prediction, at least if one averages over length scales which are at least as large as the wavelength of the most rapidly growing modes predicted by linear stability analysis.

Suppose the superadiabaticity is substantially supercritical yet small ($\epsilon_{\mathrm{CI}} \ll \epsilon \ll \chi$) and consider, for simplicity, a single inertial wave mode near the wavevector of maximal amplification. This wave is weakly amplified ($\sigma \ll \omega$) but must eventually reach sufficient amplitude for non-linear effects to occur. For this particularly simple (and artificial) example, the most important non-linearity is known both theoretically (Hasselman 1967) and from laboratory experiments (McEwan & Robinson 1975) to be a subharmonic parametric instability in which the large-scale wave is spontaneously unstable to the growth of smaller-scale waves. If molecular dissipative processes (such as viscosity) are unimportant then the ultimate sink of the energy fed into the large-scale wave is the 'traumatic' disruption of the small-scale waves, either because of shear instability (Garrett & Munk 1972) or local convective overturning (Orlanski & Bryan 1969). These 'events' transport solute upward, increasing the gravitational energy of the system at the expense of the source of wave energy. Wavebreaking is most likely to occur at small wavelengths (large wavevectors), because the wave shear increases as the square of the wavevector. It leads to small-scale layering (analogous to the fine structure observed in the oceans, Stommel & Fedorov 1967).

The mixing rate can be characterized by an eddy diffusivity D_{e}, defined such that the average change in potential energy per unit time volume is $\rho D_{\mathrm{e}} N_{\mathrm{s}}^2$. If the mixing occurs on length scales much smaller than those at which the energy input occurs, and if the wave amplitudes are sufficiently small, then it is a good approximation to estimate the energy input from linear stability theory. Equating energy input to the rate of change of potential energy then gives

$$\int \sigma(\mathbf{k})\, E(\mathbf{k})\, d^3\mathbf{k} = D_{\mathrm{e}} N_{\mathrm{s}}^2 \tag{23}$$

where $E(\mathbf{k})$ is the wave kinetic energy per unit fluid mass in the wavevector interval between $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$. (Magnetic energy is unimportant.) The right-side of equation (23) is given by stellar evolution, and $\sigma(\mathbf{k})$ is given by linear stability analysis, but it remains to determine $E(\mathbf{k})$. We assume that shear instability is the primary cause of mixing, and follow the model developed by Garrett & Munk (1972) for the Earth's oceans. Similar conclusions are reached if local convective overturning (i.e. the formation of 'rotors'; see Orlanski & Bryan 1969) is assumed instead; but shear instability is more probable for the reasons discussed by Garrett & Munk.

The mean square vertical shear is $\overline{S^2}$ given by

$$\overline{S^2} = \int \overline{S^2}(\mathbf{k})\, d^3\mathbf{k}$$

$$\overline{S^2}(\mathbf{k}) \equiv 2(k^2 - k_\perp^2)^2 E(\mathbf{k})/k^2 \tag{24}$$

and the requirement for shear instability is that the shear reach a critical value S_c such that

$$N_s^2/S_c^2 = Ri_c \simeq \tfrac{1}{4} \tag{25}$$

where Ri_c is the critical value of the gradient Richardson number (*cf.* Longuet-Higgins 1969). In accord with the Garrett & Munk model, it is assumed that the number of mixing events per unit time and vertical distance is about $\exp(-\tfrac{1}{2}S_c^2/\overline{S^2})\,T_1^{-1}L_1^{-1}$, where T_1 and L_1 are the mean intervals in time and vertical distance between shear zeros. Each mixing event mixes fluid over a vertical distance of order L_1, so it follows from the definition of D_e that

$$D_e \simeq 0.01 \exp(-2N_s^2/\overline{S^2})\, L_1^2 T_1^{-1} \tag{26}$$

where the numerical factor is essentially taken from Garrett & Munk.

The form of $E(\mathbf{k})$ is determined by both $\sigma(\mathbf{k})$ and the non-linear wave interactions. We first consider the low-field ($B \ll B_{C1}$) limit, where the anisotropy of the spectrum is not great. (Although the overstability mechanism preferentially feeds modes for which $k = k_\perp$, modes at the same k but somewhat smaller $k_\perp$ are amplified at a comparable rate and also fed by non-linear processes.) If the spectrum is approximately isotropic, then $4\pi k^2 E(\mathbf{k})$ can be replaced by $\bar{E}(k)$, the kinetic energy per unit mass in the wavevector magnitude interval between k and $k + dk$. Since $\sigma(\mathbf{k})$ is peaked at $k = k_0 = (N_s/\kappa)^{1/2}$ it is reasonable to assume a power-law spectrum of the form

$$\begin{aligned} \bar{E}(k) &\sim k^{-q}, \quad k \ll k_0 \\ &\sim k^{-p}, \quad k_0 \ll k \ll k_m \end{aligned} \tag{27}$$

where $q \ll p$ and k_m represents the termination of the 'cascade' of energy from small to large wavevectors, because of wavebreaking. In the Earth's oceans, $p \simeq 2$ (Garrett & Munk 1975) which is coincidentally not far removed from the Kolmogorov spectrum of $p = 5/3$. Although the observed spectrum has been interpreted by some as being related to the ocean floor topography (Bell 1975; Armi 1978), it is probably intrinsic to the cascade process (McComas & Bretherton 1977). Assuming $q \ll 2$, equations (24) and (26) imply

$$\sigma_m k_0 \bar{E}(k_0) \simeq D_e N_s^2 \tag{28}$$

$$\overline{S^2} \simeq k_0^3 \bar{E}(k_0)(k_m/k_0)^{3-p}. \tag{29}$$

Since $T_1 \sim \pi/N_s$ and $L_1 \sim \pi/k_m$, it follows from equation (26) that

$$\frac{D_e}{\kappa} \simeq 0.01 \left(\frac{k_0}{k_m}\right)^2 \exp\left[-\left(\frac{\sigma_m}{N_s}\right)\left(\frac{k_0}{k_m}\right)^{3-p}\left(\frac{\kappa}{D_e}\right)\right]. \tag{30}$$

Since the Gaussian factor is less than unity, it follows that $k_m \lesssim 0.1(\kappa/D_e)^{1/2} k_0$. For th maximum value of k_m and for $p = 2$, we find from equation (17) that

$$\epsilon/\chi \simeq (D_e/\kappa)^{1/2}. \tag{31}$$

The same value is obtained if one maximizes ϵ/χ, subject to the constraint of equation (30).

There is another entirely distinct argument which suggests that the actual value of ϵ/χ i of this order. McComas & Bretherton (1977) point out that the rate of resonant interactioi between wave modes is quadratic in the energy density. If we suppose that the energy inpu at k_0 is balanced by 'diffusion' (i.e. cascade) of wave action to nearby wavevectors at a rat proportional to the square of the energy density, then it follows from dimensional con siderations alone that

$$\sigma_m \bar{E}(k_0) \simeq k_0^3 [\bar{E}(k_0)]^2 N_s^{-1} \tag{32}$$

which, together with equation (28), implies $\epsilon/\chi \sim (D_e/\kappa)^{1/2}$. In these circumstances, th inequality $D_e \ll \kappa$ not only ensures that $\epsilon \ll \chi$ but also that $k_0 \bar{E}(k_0) \ll N_s^2/k_0^2$, so that th wave amplitudes near $k \sim k_0$ are small and the linearity assumption implicit in equatioi (23) is satisfied.

If ϵ were much smaller than the prediction of equation (31) then equation (28) implie that $\bar{E}(k_0)$ would have to be larger. However, this would require that the rate of wave inte action exceeds the rate at which energy is fed into the system by the overstabilit mechanism, in contradiction with the assumed steady-state spectrum. Conversely, an exces sive superadiabaticity would imply that the mode growth rate exceeds the rate at whicl non-linearities can feed the energy into other wave modes, again inconsistent with a steady state spectrum. This reasoning suggests that equation (31) is not merely an upper bound fo ϵ/χ, but also a semiquantitative estimate.

In the high-field limit, the spectrum becomes essentially two-dimensional. Let k_2 be th magnitude of the wavevector component perpendicular to $\mathbf{B}$, and let $\bar{E}_2(k_2)$ be the kineti energy between k_2 and $k_2 + dk_2$, integrated over all values of the wavevector componen parallel to $\mathbf{B}$. Since only the region $\mathbf{k} \cdot \mathbf{B} \lesssim N_s$ contributes significantly to this integral, i follows that equations (28) and (29) are unaltered except that $\bar{E}(k_0)$ is replaced b $(B_{C1}/B)\,\bar{E}_2(k_0)$. The final result for ϵ/χ is therefore only different from the isotropic case i $\bar{E}_2(k_2)$ has a different power-law dependence on k_2 than that for $\bar{E}(k)$.

To complete the model, it remains to be demonstrated that there are no other sinks o wave energy which might compete with wave breaking. Firstly, consider the rate of viscou dissipation E_ν given by

$$E_\nu = \int \nu k^2 E(\mathbf{k})\, d^3\mathbf{k}. \tag{33}$$

For either the isotropic or anisotropic energy spectrum and $p = 2$, the ratio of E_ν to energ input can be easily shown to be of order ν/D_e, which is much less than unity in the situation of interest. The Ohmic dissipation rate E_λ is given by

$$E_\lambda = \int \lambda(\omega_B^2/\omega^2)\, k^2 E(\mathbf{k})\, d^3\mathbf{k}. \tag{34}$$

For an isotropic energy spectrum and $p = 2$, the ratio of E_λ to the energy input is of orde $(B/B_{C2})^2$ where

$$B_{C2} \equiv (4\pi\rho_0 N_s D_e^2/\lambda)^{1/2}. \tag{35}$$

In Section 4, B_{C2} is tabulated and $B_{C2} \gtrsim B_{C1}$ is found to be satisfied. It follows that Ohmic dissipation is not important in the low-field regime $B < B_{C1}$. In the high-field region, $E(\mathbf{k})$ is only substantial if $\omega_B^2 \lesssim N_s^2$ since neither the overstability mechanism nor non-linearities can feed modes for which $\omega^2 \sim \omega_B^2 \gg N_s^2$. It follows that the integral in equation (34) is dominated by $\omega^2 \gtrsim \omega_B^2$ and the ratio of E_λ to the energy input is at most of order λ/D_e, which is much less than unity in the situations of interest.

The propagation of internal waves out of the semiconvective region and into the radiative envelope is another sink of wave energy. Even in the unlikely event that there is no total internal reflection of waves, the energy loss by this means for mode $\mathbf{k}$ cannot exceed about $Av(\mathbf{k})\,E(\mathbf{k})$ where $v(\mathbf{k})$ is the magnitude of the group velocity, and A is the outer surface area of the semiconvective zone. This is equivalent to an energy loss per unit volume of order $E(\mathbf{k})/\tau_1(\mathbf{k})$ where $\tau_1(\mathbf{k})$ is the time for propagation of the waves across the semiconvective zone. Energy loss by propagation is negligible if $\sigma(\mathbf{k})\,\tau_1(\mathbf{k}) \gg 1$ for $\mathbf{k} \sim \mathbf{k}_0$. If H_s is the depth of the semiconvective zone, then this requirement is satisfied provided $\epsilon/\chi \sim (D_e/\kappa)^{1/2} \gg (k_0 H_s)^{-1}$. For a well-developed semiconvective zone, $H_s \sim H_p$ and this inequality is satisfied (see Section 4). In fact, only a small fraction of the modes can propagate through the region of near zero static stability at the semiconvective–radiative boundary.

The smallness of the wavelength ($\sim k_0^{-1}$) relative to the depth of the semiconvective zone also ensures that the transitions between full convection, semiconvection and pure radiation occur over small vertical distances. This justifies our neglect of the transition zones.

4 Application and conclusion

Semiconvection can arise in a variety of circumstances but we shall concentrate here on the most prevalent and most prolonged semiconvective phase in which a growing semiconvective zone is intermediate between an inner convective core and an outer radiative envelope.

The intention in this section is not to provide a detailed description of semiconvection but rather to indicate the appropriate range of parameters which characterize such stars. In Table 1, values of the relevant parameters are tabulated for four massive stars and one low-mass helium-burning star. For 15, 30 and 60 $M_\odot$ the 'conventional' semiconvective evolutions predicted by Stothers & Chin (1976) have been used, but for our purpose these differ little from the pioneering work of Schwarzschild & Härm (1958), which is our source for the 121.1 $M_\odot$ model. The 0.66 $M_\odot$ example is taken from Sweigart & Gross (1974). The 'typical'

Table 1. Parameters characterizing the semiconvective zones of stars.

	$15\,M_\odot$	$30\,M_\odot$	$60\,M_\odot$	$121.1\,M_\odot$	$0.66\,M_\odot$
κ (cm^2 s^{-1})	1×10^{10}	2×10^{10}	2×10^{10}	1.5×10^{10}	2.5×10^{4}
ν (cm^2 s^{-1})	1×10^{2}	2×10^{2}	3×10^{2}	5×10^{2}	3×10^{-2}
D (cm^2 s^{-1})	2×10^{2}	4×10^{2}	6×10^{2}	1×10^{3}	8×10^{-2}
λ (cm^2 s^{-1})	30	25	20	20	10
D_e (cm^2 s^{-1})	2.5×10^{6}	6×10^{6}	1.5×10^{7}	4×10^{7}	1×10^{2}
N_s (s^{-1})	5.5×10^{-4}	4.5×10^{-4}	4×10^{-4}	3×10^{-4}	0.03
k_0 (cm^{-1})	2×10^{-7}	1.5×10^{-7}	1.5×10^{-7}	1.5×10^{-7}	1×10^{-3}
H_s (cm)	1.2×10^{11}	2.1×10^{11}	3.6×10^{11}	4.2×10^{11}	4×10^{8}
$(k_0 H_s)^{-1}$	4×10^{-5}	3×10^{-5}	2×10^{-5}	1.5×10^{-5}	2.5×10^{-6}
n	7	5	4	7	20
B_{C1} (gauss)	1×10^{4}	1×10^{4}	9×10^{3}	6×10^{3}	9×10^{3}
B_{C2} (gauss)	5×10^{4}	1×10^{5}	2×10^{5}	5×10^{5}	2×10^{4}
ϵ^*	8×10^{-3}	8×10^{-3}	1.1×10^{-2}	1.5×10^{-2}	3×10^{-2}

parameters in the table refer to values which are an approximate mean of the range encountered throughout the convective zone and throughout its evolution. A factor of 3 inaccuracy or variation is typical and sufficiently small for our purpose.

The thermal diffusivity κ was obtained from the tabulated opacities used by the above authors, together with estimates of density, temperature and specific heat. The specific heat increases markedly as the radiative pressure increases (Clayton 1968) and this explains the weak dependence of κ on stellar mass in the massive stars, despite the strong dependence of luminosity on mass.

The viscosity ν is estimated from the formula given by Spitzer (1962, p. 146) for a non-degenerate electron gas. The solute diffusivity D is estimated from the results of Chapman & Cowling (1970, p. 179) and the magnetic diffusivity from Spitzer (1962, p. 138).

The eddy diffusivity D_e is evaluated as a global average in the following approximate way. Let ΔM be the mass that is transported upwards a distance h in time τ_s by semiconvection. From the definition (equation (23)),

$$\rho D_e N_s^2 \simeq \frac{\Delta M g h}{V_s \tau_s} \tag{36}$$

where V_s is the volume of the semiconvective region. In the massive stars, typical orders of magnitude are $\rho \sim 1\,\mathrm{g\,cm^{-3}}$, $g \sim 2\times 10^4\,\mathrm{cm\,s^{-2}}$, $\tau_s \sim 10^{14}\,\mathrm{s}$ and ΔM is 5–10 per cent of the star mass, whence $D_e \sim 10^7\,\mathrm{cm^2\,s^{-1}}$.

Estimates of N_s and $k_0 \equiv (N_s/\kappa)^{1/2}$ follow directly from the stellar models. Notice that N_s is large compared with likely rotation rates. The smallness of $k_0 H_s$ (and, similarly, $k_0 H_p$) confirms the smallness of non-Boussinesq effects (including variation of g, κ, B, etc.) and the smallness of wave propagation out of the semiconvective zone.

The value of n is estimated from equation (22). For this n, it can be easily verified that $Ra_s \gg Ra_0 \tau^{-1/2}$ so the inequality in equation (21) is satisfied and the simple step-like distribution of solute is invalid.

The estimates of the fields B_{C1} and B_{C2} follows from the definitions given by equations (18) and (35). The actual fields in the interiors of such stars are not known, and can only be bounded above by a field of order 10^7 gauss (*cf.* Parker 1974). It is interesting to note that if the field is in equipartition with the kinetic energies predicted by mixing length theory for the convective core, then $B \sim 10^4$ gauss; and equipartition in the semiconvective zone suggests $B \sim 10^3$ gauss, so the 'critical' field B_{C1} is not far removed from likely field magnitudes.

Finally, Table 1 lists the nominal estimate ϵ^* for the fractional superadiabaticity. This is defined by

$$\epsilon^* \equiv \chi (D_e/\kappa)^{1/2} \tag{37}$$

where $\chi \sim 0.5$ typically. This is the best estimate that can be made on the basis of the semi-quantitative model that we have presented. The tabulated values indicate that a fractional superadiabaticity exceeding a few per cent is highly unlikely, except possibly in either the most massive stars or in the low-mass helium-burning stars. In the massive stars, the sensitivity of the evolution to a modest superadiabaticity has not been tested, but in view of the current difficulty in determining which of the two extreme predictions (Schwarzschild–Härm or Ledoux) is most appropriate, the present model must be regarded as essentially indistinguishable from Schwarzschild–Härm. In the low-mass helium-burning stars, Sweigart & Gross (1974) estimate that $\epsilon \sim 0.02$ would not be sufficient to greatly modify the evolution, so the SH prediction is again confirmed.

As a counter example, there may be at least one situation where convection in the presence of a solute gradient requires a large superadiabaticity. This occurs in the proposed differentiation of black dwarfs or hydrogen–helium planets such as Jupiter and Saturn (Stevenson & Salpeter 1977b). These bodies are characterized by diffusivities and transport properties which do not satisfy the strong inequalities of conventional semiconvection. The proposal is that these degenerate bodies evolve into a pressure–temperature domain in which helium has limited solubility in hydrogen. Helium 'rain' forms and descends, leaving behind an inhomogeneous layer which is mildly supersaturated in helium. Heat must be transported through this inhomogeneous layer, but (unlike conventional semiconvection) the thermal diffusivity is about a factor of 10 too small to transport the heat flux along an adiabat (Stevenson & Salpeter 1977a). In this instance, solute redistribution is easy (because of rain formation) but heat redistribution is difficult since even $D_e \sim \kappa$ (i.e. $\epsilon \sim 1$) does not provide sufficient wavebreaking to transport the imposed heat flux. Under these circumstances, the Ledoux criterion may be more appropriate (see Stevenson & Salpeter 1977b, for more details).

In conclusion, most of the semiconvective situations encountered in stars are predicted to conform to the Schwarzschild–Härm criterion (in which the temperature gradient is adiabatic). The physical picture for massive stars is as follows. Despite the high static stability of the fluid, energy can be fed from the slightly superadiabatic thermal field into the kinetic energy of internal waves, because thermal diffusion greatly exceeds other microscopic processes. The energy is predominantly in wavelengths of order 10^7 cm, but non-linear interactions cause this energy to cascade down to wavelengths of order 10^5 cm where the shear instability criterion is satisfied and wavebreaking occurs. The energy from wavebreaking goes mainly into increasing the gravitational potential energy by the upward redistribution of helium. Viscous and Ohmic dissipation are unimportant. Convective overshoot at the semiconvective–convective boundary and wave propagation at the semiconvective–radiative boundary modify this picture somewhat, but only in regions much smaller than the semiconvective zone. The apparent discrepancy between the predictions of Schwarzschild–Härm and the observed distribution of supergiants (Stothers & Chin 1976) must be sought in other physical mechanisms such as mass loss.

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